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A note on a possible anomaly in the complex numbers

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Abstract In the present paper a conflict in basic complex number theory is reported. The ingredients of the analysis are Euler's identity and the DeMoivre rule for $n = 2$. The outcome is that a quadratic equation only has one single solution because one of the existing solutions gives rise to an impossibility.

Keywords Basic complex number theory · Euler's identity and the DeMoivre rule · conflicting solution

Mathematics Subject Classification (2010) MSC 00A05 · MSC 03A05

1 Introduction

Despite the fact that the complex numbers are deeply researched into and are therefore widely applied, it is no luxury to every now and then look at elementary aspects of the theory. This small note tries to establish whether the complex numbers are consistent with all the normally in applications expected operations. It is found that perhaps there is a problem with consistency. In the paper an anomaly in elementary complex number theory [1] is presented. Only one textbook reference is presented because it is unknown to the author if other modern research into this matter exists. The author and Dr Nagata have done some research into an associated case [2]. It is unknown if this case is relevant to what is found here. The author suspects that because of the phasor $e^{i\phi(x,t)}$ in Feynman's path integral formulation of the quantum mechanics [3], [4], the results of the present small case study will have consequences for quantum mechanics.

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2 Complex number anomaly

In elementary complex number theory [1] there are two basic principles that will be employed here. The first is Euler's identity. This is $\forall_{t\in\mathbb{R}}\ e^{it} = \cos(t) + i\sin(t)$. The second is the power rule of DeMoivre. This is, $\forall_{n\in\mathbb{N}}\ (\cos(x) + i\sin(x))^n = \cos(nx) + i\sin(nx)$. Here we will use the easy to be verified form for $n = 2$.

Now let us look at the following expression for $\varphi \in \mathbb{R}$ and $\psi \in \mathbb{R}$.

$$z_{\varphi,\psi} = z = \exp\left[i(\varphi + \psi)^2\right] \tag{1}$$

Hence, [1, p 68], for any $u \in \mathbb{C}$ and $w \in \mathbb{C}$, $\exp[(u + w)] = \exp(u)\exp(w)$. We therefore may write

$$z = \exp\left[i(\varphi^2 + \psi^2)\right]\exp\left[2i\chi\right] \tag{2}$$

and $\chi =_{def} \varphi\psi$. Let us, subsequently, look at $\varphi + \psi = \sqrt{\pi}$. According to (1) $z = e^{i\pi} = -1$. Moreover, if $\alpha =_{def} \varphi\,(\varphi - \sqrt{\pi})$ then, via $\psi = \sqrt{\pi} - \varphi$

$$\begin{aligned} \chi &= -\alpha \\ \varphi^2 + \psi^2 &= \pi + 2\alpha \end{aligned} \tag{3}$$

Note we may take $\varphi \not\equiv 0$. From (1) and (2) and $z = -1$ it follows that

$$\exp\left[-2i\chi\right] = -\exp\left[i(\varphi^2 + \psi^2)\right] \tag{4}$$

The left hand of the previous equation (4) can be written according to Euler's identity as

$$\exp\left[-2i\chi\right] = \cos(2\chi) - i\sin(2\chi) \tag{5}$$

According to DeMoivre we then are allowed to write

$$\exp\left[-2i\chi\right] = \left(\cos(\chi) - i\sin(\chi)\right)^2 \tag{6}$$

Note that our starting point is a quadratic expression. The right hand of (4) we define $\beta =_{def} \frac{1}{2}(\varphi^2 + \psi^2)$ and then note that DeMoivre and Euler's identity gives

$$\exp\left[i(\varphi^2 + \psi^2)\right] = \left(\cos(\beta) + i\sin(\beta)\right)^2 \tag{7}$$

If we then define for ease of presentation the short-hand $z_\chi^2 =_{def} \exp\left[-2i\chi\right]$ together with $b_\beta^2 =_{def} \exp\left[i(\varphi^2 + \psi^2)\right]$, then, looking at the previous two equations and (4), we have an equality

$$z_\chi^2 = -b_\beta^2 \tag{8}$$

Let us subsequently define $\eta \in \{-1, 1\}$ and note that (8) must have two solutions for z_χ. We started out with $\exp[-2i\chi]$ and that is a genuine quadratic form. They are for $\eta = 1$ and for $\eta = -1$,

$$z_\chi(\eta) = i\eta b_\beta \tag{9}$$

The further explanation employs η_1,η_2 and η_3 all in $\{-1,1\}$. With $z_\chi = \eta_1 (\cos(\chi) - i\sin(\chi))$ squared on the right hand of (6) and $b_\beta = \eta_2 (\cos(\beta) + i\sin(\beta))$ squared on the right hand of (7). In addition we have $\eta_3 i$ because $(\eta_3 i)^2 = -1$. The η in (9) therefore is: $\eta = \eta_1\eta_2\eta_3$. In order to be perfectly clear on what we are looking at in equation (9) we ask the reader to consider

$$\{\eta_1 (\cos(\chi) - i\sin(\chi))\}^2 = \{\eta_3 i\}^2 \times \{\eta_2 (\cos(\beta) + i\sin(\beta))\}^2 \tag{10}$$

This equation is *exactly* the same as the equation in (8). And so it can be rightfully concluded that,

$$\cos(\chi) - i\sin(\chi) = i\eta (\cos(\beta) + i\sin(\beta)) \tag{11}$$

with, $\eta = \eta_1\eta_2\eta_3$.

Obviously, looking at (10), the $z_\chi^2 = \exp[-2i\chi]$ is only an intermediate definition, a short-hand. It is obvious that the short-hand $z_\chi^2 = \exp[-2i\chi]$ *in no way* leads to the requirement of solving an additional equation. The short-hand is for the purpose of discussing the absence of weakness of the equation (11).

Furthermore, and also looking at (5)-(7), we expect *two different* complex solutions here. They are, $z_u = (\cos(u), \sin(u))$ and $z_v = (\cos(v), \sin(v))$ and the u and v correspond to the respective η values in $\{-1,1\}$. For bracket notation viz. [1]. The values of the η_m, with $m = 1,2,3$, coefficients under study can be viewed as an $\exp[ik_m\pi]$ term in the complex number under consideration. For each m we have $k_m \in \{0,1\}$. There is, obviously, no need to extend k_m beyond the set $\{0,1\}$. Key is that the final η is only in $\{-1,1\}$ and that both η values are expected to be associated to a solution $z_u = (\cos(u), \sin(u))$ and $z_v = (\cos(v), \sin(v))$.

From the definition of χ in (3) the left hand of (11) gives rise to

$$\cos(\alpha) + i\sin(\alpha) = i\eta (\cos(\beta) + i\sin(\beta)) \tag{12}$$

From the definition of $\beta = \frac{1}{2}(\varphi^2 + \psi^2)$ and (3) it follows, $\beta = \frac{\pi}{2} + \alpha$. Therefore,

$$\begin{aligned} \cos(\beta) &= \cos\left(\frac{\pi}{2} + \alpha\right) = -\sin(\alpha) \\ \sin(\beta) &= \sin\left(\frac{\pi}{2} + \alpha\right) = \cos(\alpha) \end{aligned} \tag{13}$$

Using the above reformulations and (12) gives $-i\cos(\alpha) + \sin(\alpha) = \eta(-\sin(\alpha) + i\cos(\alpha))$. This implies

$$-i\cos(\alpha) + \sin(\alpha) = i\eta\cos(\alpha) - \eta\sin(\alpha) \tag{14}$$

Comparing real and imaginary components the result looks like

$$\begin{aligned} -\cos(\alpha) &= \eta\cos(\alpha) \\ \sin(\alpha) &= -\eta\sin(\alpha) \end{aligned} \tag{15}$$

If, $\eta = -1$ the relations in (15) can be true. However, because (8) also has a solution with $\eta = 1$, we then see that (15) cannot be satisfied. It is by definition impossible to have finite $\alpha \in \mathbb{R}$ with $\cos(\alpha) = \sin(\alpha) = 0$. This impossible result for $\eta = 1$ represents an anomaly in the complex numbers.

3 Conclusion & discussion

It is demonstrated here that an anomaly in complex numbers is derived from Euler's identity and the DeMoivre rule for $n = 2$. This may hold consequences for applications such as quantum mechanics.

The starting point of the analysis was $z_{\varphi,\psi} = \exp\left[i(\varphi+\psi)^2\right]$ and $\varphi \in \mathbb{R}$ and $\psi \in \mathbb{R}$, with $\varphi + \psi = \sqrt{\pi}$. We defined $\chi = \varphi\psi$ and obtained a relation between $z_\chi = \eta_1 (\cos(\chi) - i\sin(\chi))$ and $b_\beta = \eta_2 (\cos(\beta) + i\sin(\beta))$, both η_1 and η_2 in $\{-1, 1\}$.

An important point is that the derivation *starts* with an equivalent of $z_\chi^2 = -b_\beta^2$. This is obtained with the use of Euler's identity and the DeMoivre rule for $n = 2$ from $z_{\varphi,\psi}$. The two roots of $z_\chi^2 = -b_\beta^2$, are there *from the start* and are not introduced later on, thereby making $z_\chi = i\eta b_\beta$ in (11) a weak relation. The reader easily can verify that we did not have an equation (1) equivalent to e.g. $z_\chi = ib_\beta$ and then later on derived $z_\chi^2 = -b_\beta^2$ so that going back, there would be two roots from the original one root starting point. It can be verified that we started with an equivalent of $z_\chi^2 = -b_\beta^2$ and *then* derived a missing & conflicting root. The equation $z_\chi = i\eta b_\beta$ is therefore *not* a weak equation. The missing & conflicting root is demonstrated by derivation and is not an artefact introduced by procedure.

References

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